ECE317 : Feedback and Control

Lecture :
Stability
Routh-Hurwitz stability criterion

Dr. Richard Tymerski
Dept. of Electrical and Computer Engineering
Portland State University

Course roadmap

Modeling

- Laplace transform
- Transfer function
- Block Diagram
- Linearization
- Models for systems
  - electrical
  - mechanical
  - example system

Analysis

- Stability
  - Pole locations
  - Routh-Hurwitz
- Time response
  - Transient
  - Steady state (error)
- Frequency response
  - Bode plot

Design

- Design specs
- Frequency domain
- Bode plot
- Compensation
- Design examples

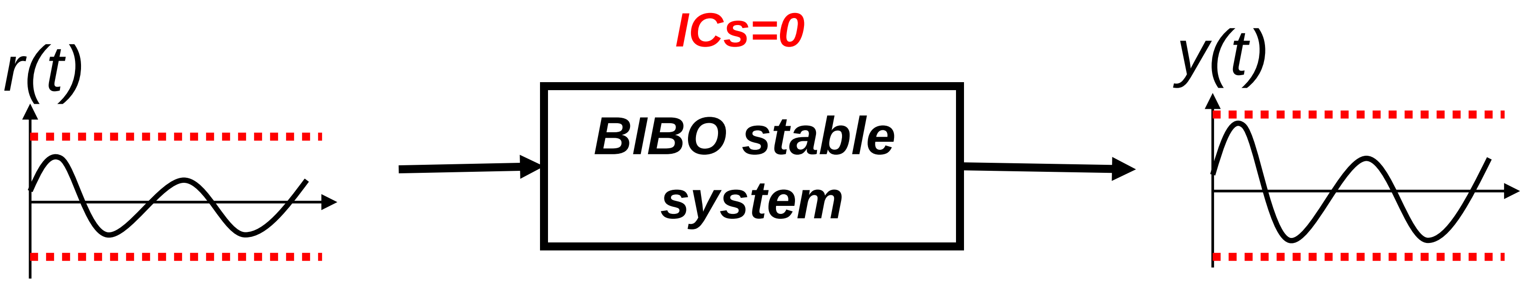
Matlab & PECS simulations & laboratories

Stability

- Utmost important specification in control design!
- Unstable systems have to be stabilized by feedback.
- Unstable closed-loop systems are useless.
- What if a system is unstable? ("out-of-control")
  - It may hit mechanical/electrical "stops" (saturation).
  - It may break down or burn out.
  - Signals diverge.
- Examples of unstable systems
  - Tacoma Narrows Bridge collapse in 1940
  - SAAB Gripen JAS-39 prototype accident in 1989
  - Wind turbine explosion in Denmark in 2008

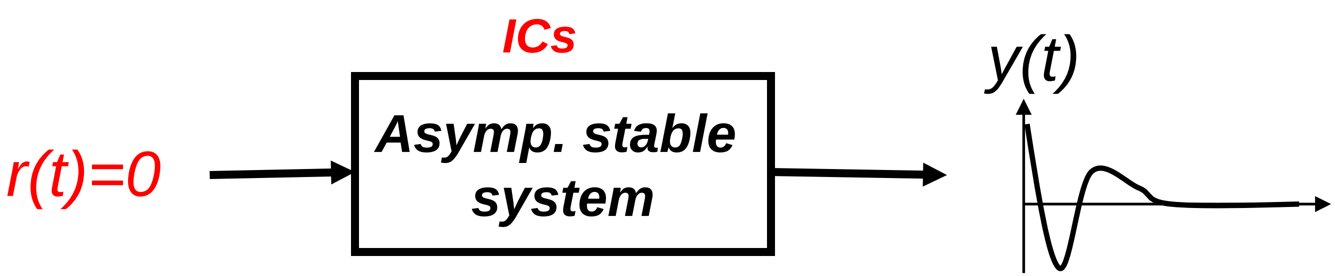
Definitions of stability

- **BIBO** (Bounded-Input-Bounded-Output) **stability**
  Any bounded input generates a bounded output.



- **Asymptotic stability**

  Any ICs generates y(t) converging to zero.



Some terminologies

$$G(s) = \frac{n(s)}{d(s)}$$

Ex. $$G(s) = \frac{(s-1)(s+1)}{(s+2)(s^2+1)}$$

- Zero: roots of *n(s)*

$$(\text{Zeros of } G) = \pm 1$$

- Pole: roots of *d(s)*

$$(\text{Poles of } G) = -2, \pm j$$

- Characteristic polynomial: *d(s)*
- Characteristic equation: *d(s)=0*

Stability condition in *s*-domain
(Proof omitted, and not required)

- For a system represented by transfer function *G(s)*,

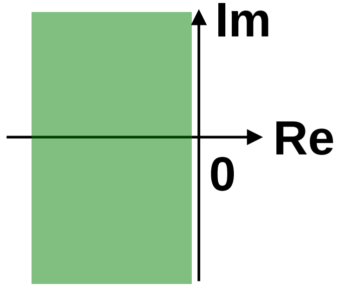
System is BIBO stable

⇕

All the poles of G(s) are in the open left half of the complex plane.

⇕

System is asymptotically stable



Idea of stability condition

- Example $$y'(t) + \alpha y(t) = r(t)$$

$$\Rightarrow \quad sY(s) - y(0) + \alpha Y(s) = R(s)$$

$$\Rightarrow \quad Y(s) = \frac{1}{s+\alpha}(R(s) + y(0))$$

Asym. Stability:
(r(t)=R(s)=0)

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+\alpha}y(0)\right\} = e^{-\alpha t}y(0) \to 0 \Leftrightarrow \mathrm{Re}(-\alpha) < 0$$

BIBO Stability:
(y(0)=0)

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{G(s)R(s)\} = \int_0^t g(\tau)r(t-\tau)d\tau = \int_0^t e^{-\alpha\tau}r(t-\tau)d\tau$$

$$|y(t)| \leq \int_0^t |e^{-\alpha\tau}||r(t-\tau)|d\tau \leq \underline{\int_0^t |e^{-\alpha\tau}|d\tau} \cdot r_{max}$$

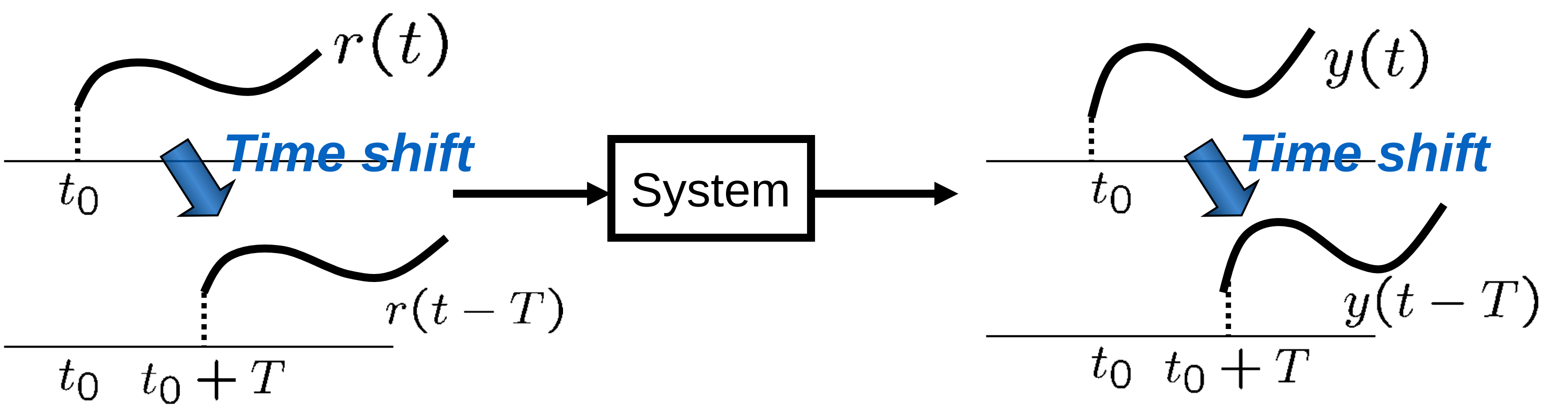
Bounded if $Re(-\alpha)<0$

Remarks on stability

- For general systems (nonlinear, time-varying), BIBO stability condition and asymptotic stability condition are different.
- For linear time-invariant (LTI) systems (to which we can use Laplace transform and we can obtain transfer functions), these two conditions happen to be the same.
- In this course, since we are interested in only LTI systems, we use simply "stable" to mean both BIBO and asymptotic stability.

Time-invariant & time-varying

- A system is called *time-invariant (time-varying)* if system parameters do not (do) change in time.
- Example: $Mx''(t)=f(t)$ & $M(t)x''(t)=f(t)$
- For time-invariant systems:



- This course deals with time-invariant systems.

Remarks on stability (cont'd)

- Marginally stable if
  - *G(s)* has no pole in the open RHP (Right Half Plane), and
  - *G(s)* has at least one simple pole on *jw*-axis, and
  - *G(s)* has no multiple pole on *jw*-axis.

$$G(s) = \frac{1}{s(s^2+4)(s+1)^2}$$

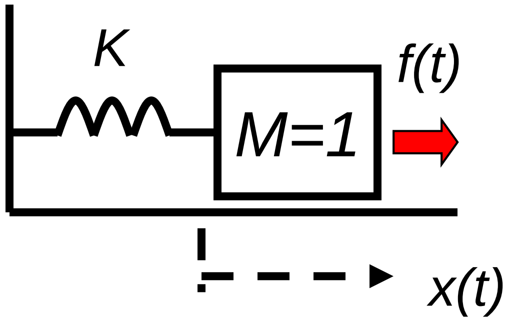
Marginally stable

$$G(s) = \frac{1}{s(s^2+4)^2(s+1)^2}$$

NOT marginally stable

- Unstable if a system is neither stable nor marginally stable.

“Marginally stable” in *t*-domain



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + K}$$

- For any bounded input, except only special sinusoidal (bounded) inputs, the output is bounded.
  - In the example above, the special inputs are in the form of:

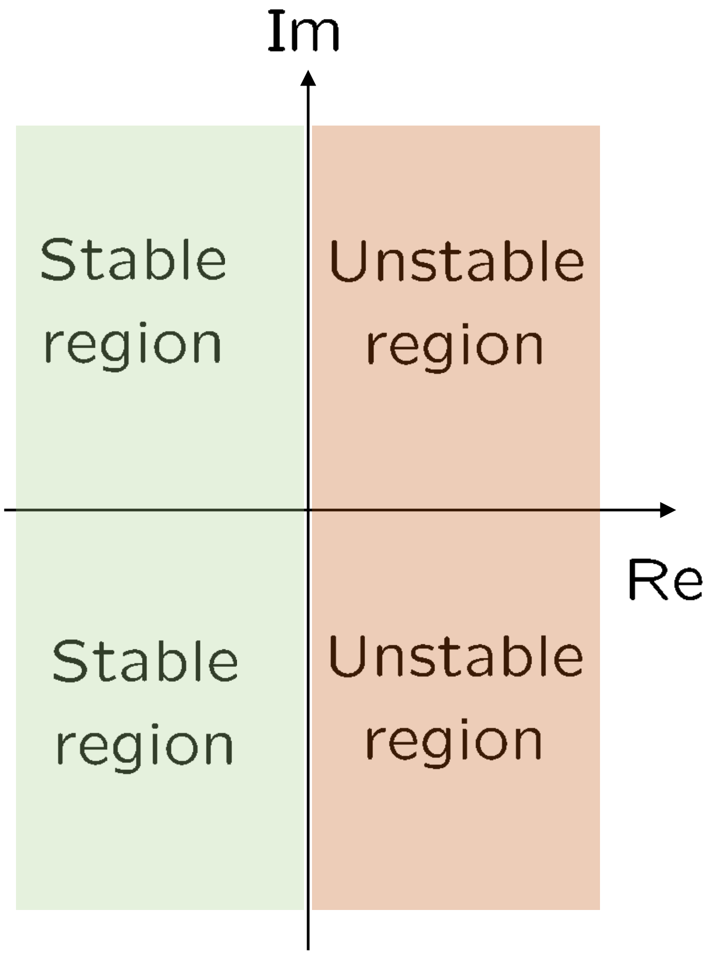
$$f(t) = \alpha \sin \sqrt{K}t + \beta \cos \sqrt{K}t \quad \Rightarrow \quad x(t) \rightarrow \pm\infty$$

- For any nonzero initial condition, the output neither converge to zero nor diverge.

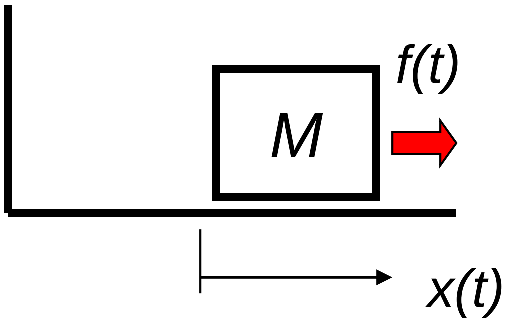
Stability summary

Let s_i be poles of $G(s)$.
Then, $G(s)$ is …

- (BIBO, asymptotically) stable if $Re(s_i)<0$ for all i.
- marginally stable if
  - $Re(s_i)<=0$ for all i, and
  - simple pole for $Re(s_i)=0$
- unstable if it is neither stable nor marginally stable.

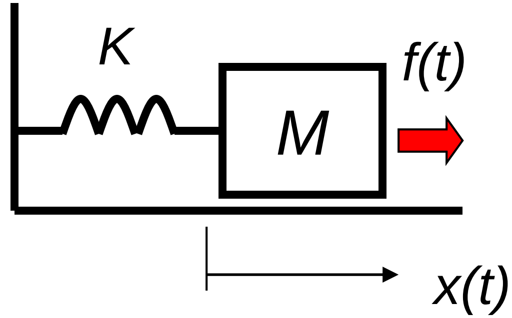


Mechanical examples



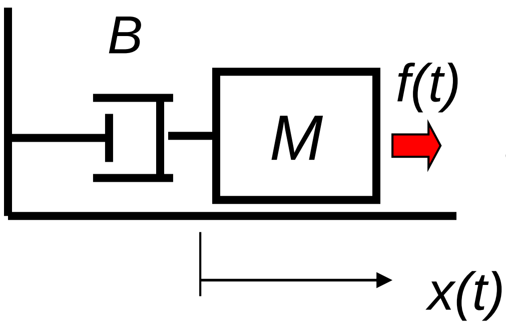
$$\frac{X(s)}{F(s)} = \frac{1}{s^2}$$

Poles=
stable?



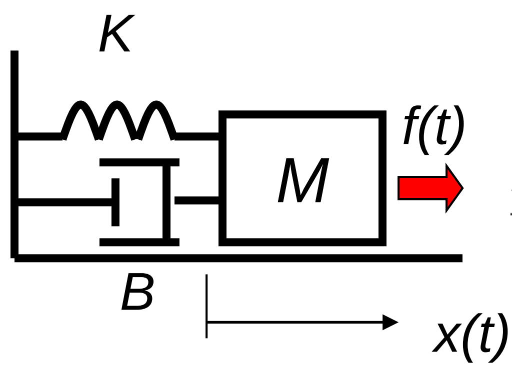
$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + K}$$

Poles=
stable?



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + Bs}$$

Poles=
stable?



$$\frac{X(s)}{F(s)} = \frac{1}{s^2 + Bs + K}$$

Poles=
stable?

Examples

$G(s)$	**Stable/marginally stable /unstable**
$\frac{20}{(s+1)(s+2)(s+3)}$	**?**
$\frac{20(s+1)}{(s-1)(s^2+2s+3)}$	**?**
$\frac{10(s-1)e^{-s}}{(s+5)(s^2+3)}$	**?**
$\frac{1}{(s+5)(s^2+2)^2}$	**?**
$\frac{1}{s^4+5s^3+10s^2+3s+1}$	**???**

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Routh-Hurwitz criterion

- This is for LTI systems with a *polynomial* denominator (without sin, cos, exponential etc.)
- It determines if all the roots of a polynomial
  - lie in the open LHP (left half-plane),
  - or equivalently, have negative real parts.
- It also determines the number of roots of a polynomial in the open RHP (right half-plane).
- It does NOT explicitly compute the roots.
- No proof is provided in any control textbook.

Polynomial and an assumption

- Consider a polynomial

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

- Assume $a_0 \neq 0$
  - If this assumption does not hold, Q can be factored as

$$Q(s) = s^m \underbrace{(\hat{a}_{n-m} s^{n-m} + \cdots + \hat{a}_1 s + \hat{a}_0)}_{\hat{Q}(s)}$$

  where $\hat{a}_0 \neq 0$

  - The following method applies to the polynomial $\hat{Q}(s)$

Routh array

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & k_1 & k_2 & & & \\
s^1 & l_1 & & & & \\
s^0 & m_1 & & & &
\end{array}
$$

From the given polynomial

$$Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$$

Routh array
(How to compute the third row)

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

$$b_1 = \frac{a_{n-2}a_{n-1} - a_n a_{n-3}}{a_{n-1}}$$

$$b_2 = \frac{a_{n-4}a_{n-1} - a_n a_{n-5}}{a_{n-1}}$$

$$\vdots$$

Routh array
(How to compute the fourth row)

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

$$c_1 = \frac{a_{n-3}b_1 - a_{n-1}b_2}{b_1}$$

$$c_2 = \frac{a_{n-5}b_1 - a_{n-1}b_3}{b_1}$$

$$\vdots$$

Routh-Hurwitz criterion

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	b_4	$\cdots$
s^{n-3}	c_1	c_2	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$			
s^2	k_1	k_2			
s^1	l_1				
s^0	m_1				

The number of roots
in the open right half-plane
is equal to
the number of sign changes
in the first column of Routh array.

Example 1

$$Q(s) = s^3 + s^2 + 2s + 8 \ (= (s+2)(s^2 - s + 4))$$

Routh array

s^3	1	2
s^2	1	8
s^1	-6	
s^0	8	

$$\frac{2-8}{1}$$

$$\frac{8 \times (-6) - 0}{-6}$$

Two sign changes
in the first column
$1 \rightarrow -6 \rightarrow 8$

Two roots in RHP

$$\frac{1}{2} \pm \frac{j\sqrt{15}}{2}$$

Example 2

$$Q(s) = s^3 + 3s^2 + 6s + 8 \ (= (s+2)(s^2+s+4))$$

Routh array

$$\begin{array}{c|cc} s^3 & 1 & 6 \\ s^2 & 3 & 8 \\ s^1 & 10/3 & \\ s^0 & 8 & \end{array} \qquad \frac{6 \cdot 3 - 8}{3}$$

Always same!

No sign changes in the first column

$$1 \rightarrow 3 \rightarrow 10/3 \rightarrow 8$$

No roots in RHP

$$-2, -\frac{1}{2} \pm \frac{j\sqrt{15}}{2}$$

Example 3 (from slide 14)

$$Q(s) = s^4 + 5s^3 + 10s^2 + 3s + 1$$

Routh array

s^4	1	10	1
s^3	5	3	
s^2	47/5	1	
s^1	(positive)		
s^0	1		

Always same!

No sign changes in the first column → No roots in RHP

Simple important criteria for stability

- 1st order polynomial $Q(s) = a_1 s + a_0$

  All roots are in LHP $\Leftrightarrow$ a_1 and a_0 have the same sign

- 2nd order polynomial $Q(s) = a_2 s^2 + a_1 s + a_0$

  All roots are in LHP $\Leftrightarrow$ a_2, a_1 and a_0 have the same sign

- Higher order polynomial $Q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0$

  All roots are in LHP $\Rightarrow$ All a_k have the same sign

Examples

$Q(s)$	All roots in open LHP?
$3s + 5$	Yes / No
$-2s^2 - 5s - 100$	Yes / No
$523s^2 - 57s + 189$	Yes / No
$(s^2 + s - 1)(s^2 + s + 1)$	Yes / No
$s^3 + 5s^2 + 10s - 3$	Yes / No

Summary

- Stability for LTI systems
  - (BIBO, asymptotically) stable, marginally stable, unstable
  - Stability for G(s) is determined by poles of G(s).
- Routh-Hurwitz stability criterion
  - to determine stability without explicitly computing the poles of a system
- Next, examples of Routh-Hurwitz criterion